Exact results for a model of interface growth

T. J. Newman*
Department of Physics, University of Oxford, Theoretical Physics,
1 Keble Road, Oxford, OX1 3NP, United Kingdom
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We study in detail a recently proposed model of interface growth that admits an exact solution [T. J. Newman, Phys. Rev. E **49**, R2525 (1994)]. In addition to explicitly calculating the previously reported results for the interface width, we investigate the role of the temporal cutoff. We find that an inverse power of this cutoff separates two different scaling regimes in time for (substrate) dimension $d > 2$. We relate this result to the fixed point structure of a simplified version of the original model that closely resembles the Kardar-Parisi-Zhang equation, and demonstrate the existence of strong-coupling behavior in this model for intermediate times.

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I. INTRODUCTION

There has been continued interest in the nonequilibrium dynamics of interfaces over recent years [2]. The ideas fueling this interest are dynamic scaling and universality. A model which has received much attention is that of Kardar, Parisi, and Zhang (KPZ) [3] which is defined by a simple nonlinear Langevin equation for the interface height h:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta. \qquad (1)$$

This equation is applicable to interfaces which evolve through growth normal to the local interface, such as simple solid-on-solid (SOS) models [4]. Exact results are available in $d = 1$ (throughout this article, d will denote the substrate dimension, so that the dimension d' of the space in which the interface exists is $d' = d + 1$) and indicate the existence of a nontrivial asymptotic scaling regime in which the interface width $W(L,t) = L^\chi f(t/L^z)$, with $f(x) \sim x^\beta$ for $x \ll 1$ with $\beta = \chi/z$, and $f(x) \to$ const for $x \gg 1$; with $\chi = 1/2$ and $z = 3/2$ [5]. The situation is less clear in the more physically relevant case of $d = 2$—the (lower) critical dimension for the model. Here the coupling λ is marginally relevant and the renormalization group (RG) flow indicates the existence of a strong-coupling fixed point for which no results are presently available. Given the limitations of conventional analytic tools, one is driven to search for other approaches with which to attack the problem. In this paper we present just such an approach—one of exact solution of a not-too-distant relative of the problem of interest, in the hope of learning something pertinent to the original problem. The main results of this calculation were presented in a recent Rapid Communication [1].

The outline of the present paper is as follows. In Sec. II we define the model and briefly describe its potential physical application. We then make a series of transformations in order to prepare for the explicit calculations in the succeeding sections. The simplest quantity to evaluate is the evolution of the average height of the interface—this is presented in detail in Sec. III. The more complicated evaluation of the interface width is presented in Secs. IV and V for d greater or less than 2, respectively. We have split the presentation into two sections due to the nontrivial role of the temporal cutoff for $d > 2$. Section VI is dedicated to the evaluation of the average interface roughness $E(t) = \langle (\nabla h)^2 \rangle$. Comparing this quantity to the time derivative of the average height indicates the relevance of the nonlinear term in the dynamics.

Another interesting quantity to calculate is the height-height correlation function $C(\mathbf{r},t) = \langle [h(\mathbf{r},t) - h(\mathbf{0},t)]^2 \rangle$ which interpolates between the interface width and the interface roughness. The evaluation of this function for the present model is of a much higher order of difficulty than for the other functions considered here, and is therefore beyond the scope of the present work.

From the results of the exact calculation we can understand which terms in the original model are relevant and thus produce a simplified version of the model. This is studied in Sec. VII. It will be seen that this simplified model may in turn be transformed into a different interface model, closely resembling the original KPZ equation. The fixed point structure of this new model will be discussed in the light of the known results obtained in the earlier sections, with emphasis being placed on an intermediate-time strong-coupling regime occurring for $d > 2$. We close with a discussion of the results in Sec. VIII.

II. DEFINITION AND "REFORMULATION" OF THE MODEL

The model is defined by the Langevin equation for the interface height $h(\mathbf{x}, t)$,

*Present address: Institüt für Theoretische Physik, Universität zu Köln, Zülpicher Str., 77, D-50937 Köln, Germany.

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$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \frac{2\nu}{\lambda}\eta \exp(-\lambda h/2\nu) \ , \qquad (2)$$

where h is an absolute height in contrast to the field h appearing in Eq. (1) which is only defined up to some additive constant (usually taken to be the mean height of the interface). This difference in interpretation for the two fields occurs because h appears in the noise term in Eq. (2) without being acted upon by some differential operator, thus breaking the translational invariance of the equation in the direction perpendicular to the substrate. The second difference is in the allowed range of the stochastic source η. In the KPZ equation, the noise is taken to be Gaussian distributed with δ-function correlations in both space and time—this is nearly always the simplest calculational choice and one relies upon some degree of universality in making such a simplification. We are unable to use this type of noise in the above equation for the following reason. We notice that the multiplicative factor becomes exponentially large for $h < 0$. This is physically unreasonable and should be avoided. To achieve positive values of the field $h(\mathbf{x}, t)$ we require the distribution $P[\eta]$ to only supply positive values of the noise η. The simplest choice turns out to be an exponentially decaying distribution (with $\eta \in [0, \infty]$)

$$P[\eta] \sim \exp -(1/D) \int d^d y \int_0^\infty dt \ \eta(\mathbf{y}, t) \ . \qquad (3)$$

All the results in this article will be calculated with such a distribution. Averages over P will be denoted by angle brackets. We have repeated the calculations for other choices of distribution, namely, uniform and bimodal, in order to address the question of universality with respect to the choice of P. The results obtained using these distributions differ from those presented here only in numerical prefactors, thus indicating a large degree of universality in these results with respect to the choice of noise distribution.

Before proceeding with the analytic investigation of Eq. (2) we shall briefly consider the physical meaning of the equation. The deterministic part of the equation is identical to the usual KPZ terms. The physical interpretation of the noise term in Eq. (2) is less clear. First one must admit that the use of Langevin equations at this phenomenological level is lacking a formal grounding. The past successes of such an approach (for instance, in phase separation [6] or fluid turbulence [5]) gives us some confidence in capturing the essential physics. The use of multiplicative noise is especially difficult to justify due to the accompanying interpretation problems (Ito versus Stratonovich) [7]. With this in mind we shall not expend too much effort in justifying the noise term in the above equation (its particular form is chosen to allow an exact solution). We may imagine that such a term would be appropriate for the situation of a KPZ-type interface evolving into an environment whose density decreases exponentially with height. If we think of a solid growing into a depositing vapor, it is clear that the noise strength will be crudely proportional to the density of the vapor. Therefore, if the density of the vapor decreases exponentially with increasing height, the noise term in Eq. (2) is not unreasonable.

Objections to the preceding discussion in defence of the physical status of the equation may certainly be raised. However, we would like to stress that the main reason for studying Eq. (2) is to learn something about real interfaces (as described by the KPZ equation, for instance), and the model above has been carefully chosen for just this purpose as we shall soon see.

Before proceeding with the calculation we may gain some insight into the likely behavior of the system from simple physical arguments. Imagine an initial condition of a flat interface located at $h = 0$. For early times, the size of $\lambda h/2\nu$ will be small compared to unity and we may expand the exponential factor in the noise. The leading order equation in this limit is simply the Edwards-Wilkinson (EW) model [8] (linear diffusion equation for h) and we may expect to retrieve the usual results obtained for that model—i.e., the average height will proceed linearly in time (the noise has nonzero mean), and the width will increase as $W(t) \sim t^{(2-d)/4}$ for $d < 2$, as $[\ln(t)]^{1/2}$ for $d = 2$, and will rapidly grow to a constant (depending on the temporal cutoff) for $d > 2$. As time proceeds, the exponential factor in the noise will become relevant and the effective noise strength will be severely reduced. Exact prediction is now difficult from an intuitive level, but we can expect the average height to increase logarithmically slowly (thus preserving the rate of deposition as h increases), and the interface width to decrease in time, due to the strongly reduced noise strength. We shall find that these simple predictions are indeed qualitatively correct, thus indicating that the physical contact with the KPZ equation and related models is lost, since in these models one expects growing fluctuations in time. However, in Sec. VII, we shall see how to apply these results to a related model which is much more closely related to the original KPZ equation.

As a first step, we make the simple rescaling $h \to h = \lambda h/2\nu$, so that the field h is now dimensionless. Equation (2) is now of the form

$$\partial_t h = \nu \nabla^2 h + \nu(\nabla h)^2 + \eta e^{-h}. \qquad (4)$$

The scaling of h to a dimensionless variable has removed λ from the equation of motion, therefore λ is no longer a free parameter in the model. However, in Sec. VII we shall study a simplified version of our model which is the natural "bare" theory, analogous to the bare theory ($\lambda = 0$) for the KPZ equation which is the EW model.

The Hopf-Cole transformation $w = e^h$ when applied to Eq. (4) yields the linear diffusion equation for w:

$$\partial_t w = \nu \nabla^2 w + \eta. \qquad (5)$$

The same transformation may be applied to the KPZ equation and one then obtains a multiplicative noise equation for w which may be formally interpreted (most clearly by use of the Feynman-Kac formula) as describing the evolution of the generating function for directed walks in a random medium [9]. No such interpretation exists for the present model.

For some initial condition $w(\mathbf{x}, 0)$ we have the solution of Eq. (5) in terms of the heat kernel $g(\mathbf{x}, t) =$

$(4\pi\nu t)^{-d/2}\exp(-x^2/4\nu t)$ in the form

$$w(\mathbf{x},t) = \int d^d y \int_0^t dt'\, g(\mathbf{x}-\mathbf{y},t-t') \times \left[w(\mathbf{y},0)\delta(t') + \eta(\mathbf{y},t')\right]. \quad (6)$$

Inverting our original transformation then yields the exact solution of Eq. (4) as

$$h(\mathbf{x},t) = \ln\left\{ \int d^d y \int_0^t dt' g(\mathbf{x}-\mathbf{y},t-t') \times \left\{\exp[h(\mathbf{y},0)]\,\delta(t') + \eta(\mathbf{y},t')\right\}\right\}. \quad (7)$$

In what follows we shall only consider the case of an initially flat interface located at $h=0$, so that $h(\mathbf{x},0)=0$ for all $\mathbf{x}$. More general initial conditions may be handled within this approach and such an extended study would be of interest. We also mention that the above transformation yields an exact solution as demonstrated above in the presence of an arbitrary function of time $f(t)$ appearing as an additional multiplicative factor in the noise term of Eq. (4). The case of $f(t)=e^{ct}$ is of interest since one can show that the average height proceeds linearly in time with a velocity $2\nu c/\lambda$. It was our original intention to include a study of the fluctuations in this case in the present article. However, we feel that it would be better presented elsewhere due to the more general results and issues to be presented here.

(Before proceeding, it is worth commenting that the range of possible transformations similar to the Hopf-Cole transformation above is vast, and that among this great number there may be some of genuine interest. The general action of such transformations may be understood as to take linear, additive noise Langevin equations (for which one knows the solution) into nonlinear, multiplicative noise equations. The difficulties in proceeding are twofold. First, the resulting nonlinear equation should have some relation to physics in order to be worthy of study [since an infinite number may be generated trivially from Eq. (5)]; and second, the transformation used should be invertible in order to express the solution of the nonlinear equation in terms of the Green's function solution of the original linear problem.)

With our choice of initial condition the solution of Eq. (4) is given by

$$h(\mathbf{x},t) = \ln\left\{1 + \int d^d y \int_0^t dt'\, g(\mathbf{x}-\mathbf{y},t-t')\eta(\mathbf{y},t')\right\}. \quad (8)$$

We are of course interested not in the solution of Eq. (4) as given above, but in averaged properties of it over the distribution P. In particular we shall examine the average height $\langle h\rangle$, the interface width $W(t) = [\langle h^2\rangle - \langle h\rangle^2]^{1/2}$ and the average interface roughness $E(t) = \langle(\nabla h)^2\rangle$. From the form of $h(\mathbf{x},t)$ given above in Eq. (8) we see that the evaluation of such quantities necessitates the averaging over a logarithm of a space-time integral of the noise η. Much of our progress with this problem is made possible with the use of the following representation of the logarithm function

$$\ln z = \int_0^\infty \frac{du}{u}(e^{-u} - e^{-uz}) \quad (9)$$

which was introduced in a previous study of deterministic Burgers turbulence [10,11]. One sees that this representation pushes the argument of the logarithm into the argument of the exponential function, which for averaging purposes is very natural. Combining Eq. (8) with Eq. (9) yields

$$h(\mathbf{x},t) = \int_0^\infty \frac{du}{u} e^{-u}[1-\psi(u)]\ , \quad (10)$$

where

$$\psi(u) = \exp\left(-u\int d^d y \int_0^t dt'\, g(\mathbf{x}-\mathbf{y},t-t')\eta(\mathbf{y},t')\right). \quad (11)$$

This ends the preparatory steps required in order to begin the calculation proper. We start with the simplest calculation, that of the average height, in the next section.

III. EVALUATION OF THE AVERAGE HEIGHT

The calculation to be presented in this section is, in essence, quite simple. However, since it sets the stage for most of what is to follow, we shall present the derivation of $\langle h\rangle$ in some detail. Since we assume translational invariance in the substrate (hyper) plane, we may content ourselves with calculating the average height at the origin of spatial coordinates $\mathbf{x}=\mathbf{0}$. From Eq. (10) we have

$$\langle h(\mathbf{0},t)\rangle = \int_0^\infty \frac{du}{u} e^{-u}[1-\langle\psi(u)\rangle]. \quad (12)$$

With the distribution defined in Eq. (3) one needs to evaluate the following path integral for $\langle\psi(u)\rangle$ [the use of the term "path integral" really denotes a multiple (discretized) space-time integration over the probability function P; possible misuse of the term due to the nonexistence of continuous paths in η space is acknowledged]:

$$\langle\psi(u)\rangle = \prod_m \prod_{n=0}^{N-1} \frac{a^d\Delta}{D}\int_0^\infty d\eta_{m,n} \times \exp\left(-a^d\Delta\eta_{m,n}[(1/D) + u g_{m,N-n}]\right) = \prod_m \prod_{n=1}^{N} [1+uDg_{m,n}]^{-1}. \quad (13)$$

A few comments are in order here concerning notation. We have discretized space using a cutoff a and have defined an integer site label $m = y_i/a$. Time has also been discretized using a cutoff Δ and we have defined an integer time slice label $n = t'/\Delta$. The symbol $g_{m,n}$ has the obvious meaning of $g(\mathbf{y}/a, t'/\Delta)$ with the functional form of g as defined earlier in Sec. II. The limits of the product integration over time are from the initial time $t' = 0$ to the final time slice but one, $N = t/\Delta - 1$, the final time slice being excluded from the integration, due to adopting an essentially Ito prescription for the noise in which the noise at a slice n_i is considered uncorrelated with the field h_{m,n_i} in that slice.

We now wish to reexponentiate the expression so that the product may be considered as a sum, and a continuum limit (i.e., replacing sums with integrals according to $a^d\Delta\sum_m\sum_n \to \int d^dy \int dt'$) may be taken. Following these steps leads us to

$$\langle\psi(u)\rangle = \exp\left(-\frac{1}{a^d\Delta}\int d^dy\int_\Delta^t dt' \;\ln[1+uDg(\mathbf{y},t')]\right). \tag{14}$$

For ease of notation we shall now rescale all lengths with respect to a. The explicit a dependence of the results may be obtained from simple dimensional analysis. We prefer to retain the temporal cutoff explicitly, due to its appearance in the lower limit of the time integral in Eq. (14). We now wish to make some simple substitutions in order to simplify the above expression. We define $s = t'/t$ and $x = y^2/(4\nu t')$ giving (after performing the angular integrations over the substrate)

$$\langle\psi(u)\rangle = \exp\left(-\frac{\tau t}{\Delta\Gamma(d/2)}\int_0^\infty dx\; x^{d/2-1} \times\int_{\Delta/t}^1 ds\; s^{d/2}\ln[1+(uD/\tau)s^{-d/2}e^{-x}]\right), \tag{15}$$

where $\tau \equiv (4\pi\nu t)^{d/2}$ and $\Gamma(c)$ is the gamma function [12]. Finally we integrate by parts on x and obtain

$$\langle\psi(u)\rangle = \exp\Big[-\frac{uDt}{\Delta}\phi\Big(\frac{uD}{\tau},\frac{\Delta}{t}\Big)\Big]\;, \tag{16}$$

where

$$\phi(z,\delta) = [\Gamma(1+d/2)]^{-1}\int_0^\infty dx\; x^{d/2}e^{-x}Q(ze^{-x},\delta)\;, \tag{17}$$

with

$$Q(\alpha,\delta) = \int_\delta^1 ds\;\frac{s^{d/2}}{(s^{d/2}+\alpha)}\;. \tag{18}$$

The expression for $\langle\psi(u)\rangle$ is all that is required for the calculation of not only the average height, but also the width and surface roughness as well; so no more explicit averaging will be required.

The function $\phi(z,\delta)$ is rather complex. To simplify our analysis we shall concentrate on definite (asymptotic) time regimes. The essential limits we shall take are $z = (uD/\tau) \ll 1$ and $\delta = \Delta/t \ll 1$, so we are interested in the regime where both arguments of ϕ are small compared to unity. One may convince oneself that in order to calculate the leading term for the average height, one simply needs the leading term in the double expansion of ϕ in terms of z and δ. When we come to evaluate the width in the subsequent sections, the next order will be required and will be found to have a nontrivial dependence on the substrate dimension d.

The leading term in the expansion of $\phi(z,\delta)$ is simply unity, so we approximate Eq. (16) by

$$\langle\psi(u)\rangle = \exp\Big(-\frac{uDt}{\Delta}\Big). \tag{19}$$

Substitution of the above form into Eq. (12) along with the use of the logarithm representation Eq. (9) then yields

$$\langle h\rangle = \ln(1+Dt/\Delta) \tag{20}$$

for all d. When we come to study the corrections to the function ϕ we shall find that Eq. (20) is valid throughout the entire region of interest. The form for $\langle h\rangle$ indicates a natural crossover time $t_1 = \Delta/D$. For $t \ll t_1$ we have $\langle h\rangle = t/t_1$, i.e., linear growth in accordance with the qualitative prediction stated earlier. For $t \gg t_1$ we have $\langle h\rangle = \ln(t/t_1)$ which again is in accord with our previous intuitive reasoning.

In the next section we shall evaluate the interface fluctuations for $d > 2$. This is more involved than the calculation for $d \leq 2$, but we have chosen to present it first since it follows naturally from the above calculation, and also because it will indicate a much simpler way to evaluate $W(t)$ for the case of $d \leq 2$. The main physical result to appear from the following calculation will be the emergence of another crossover time t_2 which is only relevant for $d > 2$.

IV. EVALUATION OF THE INTERFACE WIDTH FOR DIMENSIONS > 2

The interface width is defined as $W(t) = [\langle h^2\rangle - \langle h\rangle^2]^{1/2}$. It involves bilinear forms of logarithms, so we apply the logarithm representation Eq. (9) twice. One then finds the following form for W:

$$W^2(t) = \int_0^\infty\frac{du}{u}\int_0^\infty\frac{dv}{v}\;e^{-(u+v)}[\langle\psi(u+v)\rangle -\langle\psi(u)\rangle\langle\psi(v)\rangle]. \tag{21}$$

Therefore knowledge of the function $\langle\psi(u)\rangle$ suffices to calculate the width.

In order to get a feel for the calculation, let us take t very large (with respect to all time scales), and expand the function $\langle\psi(u)\rangle$ for small values of u. Referring to Eq. (14) we can expand the logarithm in powers of u and perform the integrals over powers of the heat kernel. One then finds

$$\langle\psi(u)\rangle = \exp\Big\{-\frac{uDt}{\Delta}+\frac{u^2D^2}{(d-2)(8\pi\nu\Delta)^{d/2}} +O\big(u^3D^3(\nu\Delta)^{-d}\big)\Big\}. \tag{22}$$

Substituting this expression into Eq. (21) and rescaling u and v with respect to $Dt/\Delta = t/t_1$ then yields

$$W^2(t) = \int_0^\infty \frac{du}{u} \int_0^\infty \frac{dv}{v} \, e^{-(u+v)} \left\{ \exp\left[\frac{\Delta^{2-d/2}}{(d-2)(8\pi\nu)^{d/2} t^2}(u+v)^2 + O(u^3\Delta^{3-d}\nu^{-d}t^{-3}) \right] - \exp\left[\frac{\Delta^{2-d/2}}{(d-2)(8\pi\nu)^{d/2} t^2}(u^2+v^2) + O(u^3\Delta^{3-d}\nu^{-d}t^{-3}) \right] \right\} . \quad (23)$$

We see that the important values of u and v are $\sim O(1)$ due to the exponential prefactor in the integrand. We may therefore expand down the exponential terms in the curly brackets as long as $\Delta^{2-d/2}\nu^{-d/2}t^{-2} \ll 1$. Also for consistency in the expansion in u and v we require that the cubic terms are smaller than the quadratic terms, which implies $\Delta^{1-d/2}\nu^{-d/2}t^{-1} \ll 1$. These conditions translate to $t \gg \Delta^{1-d/4}\nu^{-d/4}$ and $t \gg \Delta^{1-d/2}\nu^{-d/2}$. For $\nu \ll 1/\Delta$, the second condition is the stronger, and this becomes the definition of a second crossover time:

$$t_2 = \frac{\Delta^{1-d/2}}{\nu^{d/2}} . \quad (24)$$

This calculation was performed for $d > 2$, and we see that t_2 can become very large as the effective temporal cutoff is made smaller. For $t \gg t_2$ the above expansion is valid and we have from Eq. (23) the result

$$W^2(t) = \frac{2(8\pi)^{-d/2}}{(d-2)} \frac{(\Delta t_2)}{t^2}. \quad (25)$$

In order to calculate the width for $t \ll t_2$ we need to study the function $\langle\psi(u)\rangle$ more carefully. Referring to Eqs. (16)–(18) we shall expand the function $\phi(z,\delta)$ in the regime defined by (i) $z \ll 1$, (ii) $\delta \ll 1$ with (iii) $z \gg \delta^{d/2}$. The third condition will become transparent in the course of the calculation. Let us concentrate first on $Q(\alpha,\delta)$ where $\alpha \equiv ze^{-x} \ll 1$ due to condition (i). Changing variables $s \to s = \alpha s^{-d/2}$ we have

$$\begin{aligned} Q(\alpha,\delta) &= (2/d)\alpha^{2/d} \int_\alpha^{\alpha\delta^{-d/2}} ds \, \frac{s^{-1-2/d}}{(1+s)} \\ &= (2/d)\alpha^{2/d} \int_\alpha^\infty ds \, \frac{s^{-1-2/d}}{(1+s)} \\ &\quad -(2/d)\alpha^{2/d} \int_{\alpha\delta^{-d/2}}^\infty ds \, \frac{s^{-1-2/d}}{(1+s)} \\ &\equiv Q_1(\alpha,\delta) + Q_2(\alpha,\delta) . \end{aligned} \quad (26)$$

The functions Q_1 and Q_2 can be expressed in terms of the hypergeometric function [1]. However, it suits our purpose better to analyze these functions in their integral representation. Before proceeding let us define the functions $(i = 1,2)$

$$\phi_i(z,\delta) = [\Gamma(1+d/2)]^{-1} \int_0^\infty dx \, x^{d/2} e^{-x} Q_i(ze^{-x},\delta) , \quad (27)$$

so that $\phi = \phi_1 + \phi_2$.

A. Analysis of ϕ_1

To evaluate ϕ_1 we first study $Q_1(\alpha,\delta)$. Referring to Eq. (26) and using the trivial identity $(1+s)^{-1} = 1 - s(1+s)^{-1}$, we find

$$Q_1 = 1 - \frac{2}{d}\alpha^{2/d} B(1-2/d, 2/d) + \frac{2}{d}\alpha^{2/d} \int_0^\alpha ds \, \frac{s^{-2/d}}{(1+s)}, \quad (28)$$

where $B(a,b)$ is the beta function [12]. Since we are interested in $\alpha \ll 1$, we may expand the final term in Eq. (28) in powers of α. Performing the integral over x, as defined in Eq. (27), then yields

$$\phi_1 = 1 - \frac{2}{d} z^{2/d} B(1-2/d, 2/d)(1+2/d)^{-(1+d/2)} + O(z) . \quad (29)$$

B. Analysis of ϕ_2

This function is slightly more complicated to evaluate due to condition (iii). The lower limit of the s integration in Q_2 is $\alpha\delta^{-d/2} = z\delta^{-d/2}e^{-x}$. Condition (iii) indicates that for small x this quantity is much larger than unity. However, if x is large enough this is no longer true. We therefore split the range of the x integration into two regions separated by $x_0 = \ln(z\delta^{-d/2})$. Condition (iii) indicates that $x_0 \gg 1$. We therefore have

$$\begin{aligned} \phi_2 &= [\Gamma(1+d/2)]^{-1} \int_0^{x_0} dx \, x^{d/2} e^{-x} Q_2^{(1)} \\ &\quad +[\Gamma(1+d/2)]^{-1} \int_{x_0}^\infty dx \, x^{d/2} e^{-x} Q_2^{(2)} \\ &\equiv \phi_2^{(1)} + \phi_2^{(2)} , \end{aligned} \quad (30)$$

where $Q_2^{(i)}$ is given by the integral form in Eq. (26) with the lower limit greater than (less than) unity for $i = 1$ (2).

We shall not go through the explicit evaluation of $\phi_2^{(1,2)}$. The essential point is that the x integrals are dominated by the region around x_0 allowing an asymptotic expansion in inverse powers of x_0; hence condition (iii). We find

$$\phi_2 = -\delta \frac{x_0^{d/2}}{z\delta^{-d/2}} \left[\frac{4x_0}{(d+2)^2} + A(d) + O(x_0^{-1}) \right] , \quad (31)$$

where

$$A(d) = \frac{(d-1)}{(d-2)} - \frac{2}{(d+2)} B(1-2/d, 2/d) + \frac{2}{d} \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{1}{(n+2)(n+1-2/d)} + \frac{1}{n(n+1+2/d)} \right\}. \quad (32)$$

This completes the evaluation of ϕ, in the regime described by conditions (i)–(iii).

Putting this together, we have from Eqs. (16), (21), (29), and (31) the following expression for the width:

$$W^2(t) = \int_0^\infty \frac{du}{u} \int_0^\infty \frac{dv}{v}\, e^{-(u+v)(1+Dt/\Delta)} \left\{ \exp\left[b_1(d) \frac{[D(u+v)]^{1+2/d}}{\Delta\nu} + b_2(d)(\nu\Delta)^{d/2}[x_0(u+v)]^{1+d/2}[1+O(x_0^{-1})] \right] - \exp\left[b_1(d) \frac{D^{1+2/d}(u^{1+2/d}+v^{1+2/d})}{\Delta\nu} + b_2(d)(\nu\Delta)^{d/2}[x_0(u)^{1+d/2} + x_0(v)^{1+d/2}][1+O(x_0^{-1})] \right] \right\} , \quad (33)$$

where $x_0(u) = \ln[Du/(\nu\Delta)^{d/2}]$, and $b_1(d)$ and $b_2(d)$ are functions of d which are just shorthand for the d-dependent prefactors in the functions ϕ_1 and ϕ_2. We have dropped the term in ϕ_1 which is linear in z since it is always subdominant to the nonanalytic term of order $z^{2/d}$ (this will no longer be true for $d < 2$.) Condition (iii) is equivalent to $u \gg (\nu\Delta)^{d/2}/D$. We should therefore cut off the u and v integrals at $u_0, v_0 \sim (\nu\Delta)^{d/2}/D$.

Let us consider $t \gg t_1 = \Delta/D$. We may then rescale u and v by t/t_1, so that from Eq. (33) the important values of u and v are of $O(1)$. The cutoffs u_0 and v_0 are now of order t/t_2 [where t_2 is the crossover time defined in Eq. (24)]. We therefore have

$$W^2(t) = \int_{u_0}^\infty \frac{du}{u} \int_{v_0}^\infty \frac{dv}{v}\, e^{-(u+v)} \times \left\{ \exp\left[b_1(d) \frac{\Delta^{2/d}(u+v)^{1+2/d}}{\nu t^{1+2/d}} + b_2(d)(\nu\Delta)^{d/2}[y_0(u+v)]^{1+d/2}[1+O(y_0^{-1})] \right] - \exp\left[b_1(d) \frac{\Delta^{2/d}(u^{1+2/d}+v^{1+2/d})}{\nu t^{1+2/d}} + b_2(d)(\nu\Delta)^{d/2}[y_0(u)^{1+d/2} + y_0(v)^{1+d/2}][1+O(y_0^{-1})] \right] \right\} , \quad (34)$$

where $y_0(u) = \ln(ut_2/t)$.

Let us now take the limit $t \ll t_2$. First, the cutoffs u_0 and v_0 may be taken to zero. Also, the terms in Eq. (34) denoted by $O(y_0^{-1})$ may be safely discarded. We are therefore left to compare the b_1 terms to the b_2 terms. One finds that the ratio of the b_1 terms to the b_2 terms is of order $(t_2/t)^{1+2/d}\,[\ln(t_2/t)]^{-(1+d/2)}$, and therefore the important terms are those with prefactor b_1. The actual size of these terms is $O((\Delta/t)^{2/d}(\nu t)^{-1})$ which is much less than unity (for $t \gg \Delta^{2/(d+2)}\nu^{-d/(d+2)}$), and therefore we may expand these terms down from the exponential to finally obtain

$$W^2(t) = c(d) \frac{\Delta^{2/d}}{\nu t^{(d+2)/d}} , \quad (35)$$

where

$$c(d) = \frac{2}{d}(4\pi)^{-2/d} B(1-2/d, 2/d)(1+2/d)^{-(1+d/2)} \times \Gamma(1+2/d) \int_0^\infty dv\, \frac{[(1+v)^{1+2/d} - 1 - v^{1+2/d}]}{v(1+v)^{1+2/d}}. \quad (36)$$

To summarize, Eq. (35) is the form for the interface width in the regime $t_1 \ll t \ll t_2$ for $d > 2$.

For $t \ll t_1$, one may see directly from Eq. (33) that the exponential terms labeled by b_1 dominate, and may be expanded down giving one an expression for the interface width which is simply a constant. This is consistent with one's expectations from the early-time analogy with the EW model. This completes our analysis of the interface width for $d > 2$. We have identified three important time regimes separated by the crossover times t_1 and t_2, and have calculated the dominant form of the width in each regime. Simple interpolation formulas from the $t \ll t_1$ to the $t_1 \ll t \ll t_2$ forms for $W(t)$ may be readily derived from the above analysis. The explicit crossover form for $W(t)$ around t_2 is less easy to evaluate.

The appearance of the nontrivial regime for $t_1 \ll t \ll t_2$ is related to the appearance of the nonanalytic term in the expansion of $\phi(z,\delta)$, cf. Eq. (29). The naive expansion of $\langle\psi(u)\rangle$ at the beginning of this section required the cutoff Δ to keep the terms finite. However, the expansion still broke down for $t \ll t_2$. The reason for this is the existence of the nonanalytic term of order $z^{2/d}$. In a sense this term replaces the regularization role of the parameter Δ for intermediate times. In fact, the theory is well defined for all times if one takes $\delta \to 0$ in the function ϕ, since the nonanalytic regulator controls the potential divergences of the function. We note finally that the nonanalytic term is only relevant for $d > 2$. For $d < 2$ the linear term in z dominates the behavior of $\phi(z,\delta)$. The case $d = 2$ is marginal—the expansion of ϕ involves the term $z\ln(z)$ [1] which actually wins asymptotically over the linear term. However, for $d = 2$, the window of nontrivial behavior closes since the magnitude of t_2 becomes of "microscopic" proportions.

V. EVALUATION OF THE INTERFACE WIDTH FOR DIMENSIONS ≤ 2

As indicated at the end of the preceding section, the irrelevance of the nonanalytic term in the expansion of ϕ for $d \leq 2$ means that the behavior of the interface width for these dimensions is much simpler. One now has just two dynamic regions, separated by the crossover time t_1. We shall not derive the results for $W(t)$ using the elaborate set of functions from the last section (although this is of course possible [1]), but shall content ourselves with the simple analytic expansion utilized at the beginning of the preceding section.

Referring to Eq. (14) and expanding the logarithm in powers of u, we find

$$\langle\psi(u)\rangle = \exp\left\{-\frac{uDt}{\Delta} + \frac{u^2D^2t}{(2-d)\Delta(8\pi\nu t)^{d/2}} + O(u^3D^3(\nu t)^{-d}(t/\Delta))\right\} \quad (37)$$

for $d < 2$. Substituting this expression into Eq. (21) and scaling u and v by $t_1 = \Delta/D$ (which is valid for $t \gg t_1$) yields

$$W^2(t) = \int_0^\infty \frac{du}{u}\int_0^\infty \frac{dv}{v}\, e^{-(u+v)}\left\{\exp\left[\frac{\Delta}{(2-d)(8\pi\nu)^{d/2}t^{1+d/2}}(u+v)^2 + O(u^3\Delta^2\nu^{-d}t^{-(d+2)})\right] - \exp\left[\frac{\Delta}{(2-d)(8\pi\nu)^{d/2}t^{1+d/2}}(u^2+v^2) + O(u^3\Delta^2\nu^{-d}t^{-(d+2)})\right]\right\}. \quad (38)$$

For times larger than the cutoff, we may expand the exponential to obtain

$$W^2(t) = \frac{2\Delta}{(d-2)(8\pi\nu)^{d/2}t^{1+d/2}}, \qquad t \gg t_1. \quad (39)$$

For $t \ll t_1$ one may repeat the above steps without the (u,v) rescalings to find

$$W^2(t) = \frac{2D^2t^{1-d/2}}{(d-2)(8\pi\nu)^{d/2}\Delta}, \qquad t \ll t_1. \quad (40)$$

This result is in accordance with the usual EW conjecture for this model for short times.

One may repeat the above steps for the case of $d = 2$ to find

$$W^2(t) = \frac{\Delta\ln(t/\Delta)}{(8\pi\nu)t^2}, \qquad t \gg t_1 \quad (41)$$

and the EW result

$$W^2(t) = \frac{D^2\ln(t/\Delta)}{(8\pi\nu)\Delta}, \qquad t \ll t_1. \quad (42)$$

This completes our study of the interface width for this model. Before examining a simplified model of the original equation, we shall study the interface roughness in the next section. This will increase our understanding of the results obtained in this and the preceding section.

VI. EVALUATION OF THE AVERAGED INTERFACE ROUGHNESS

The nonlinear term in the deterministic part of Eq. (4) is essential if one is to utilize the Hopf-Cole transformation in order to achieve an exact solution. However, the results obtained above indicate that for $t \gg t_1$ the fluctuations are decaying in time, which would lead one to expect that this term becomes subdominant to the linear diffusion term. In order to clarify this point, we shall calculate the average surface roughness $E(t) = \langle(\nabla h)^2\rangle$ and compare the magnitude of this (with a factor of ν) to the time derivative of the average height [13]. If the nonlinear term is relevant, it should be (at least) of the same order as $\partial_t\langle h\rangle$ which for $t \gg t_1$ has a magnitude of $1/t$, cf. Eq. (20). We shall not present this calculation in detail, since it may be constructed easily from the preceding account of the evaluation of $W(t)$.

The most convenient way to calculate $E(t)$ is through the limiting procedure

$$E(t) = \lim_{r\to 0} \frac{C(\mathbf{r},t)}{r^2} , \tag{43}$$

where $C(\mathbf{r},t) \equiv \langle [h(\mathbf{r},t) - h(\mathbf{0},t)]^2\rangle$. This correlation function may be evaluated through the double application of the logarithm representation Eq. (9) in much the same way as was performed for $W(t)$ in Eq. (21). One then is confronted with the simple, but tedious, task of taking the limit of $r \to 0$; i.e., extracting the leading r^2 dependence of $C(\mathbf{r},t)$. One finally obtains for $t \gg t_1$

$$\begin{aligned} E(t) &= \frac{\Delta}{4\nu t^2}(8\pi\nu)^{-d/2} \int_\Delta^t dt'\ t'^{-(1+d/2)} \\ &= \left(\frac{(8\pi)^{-d/2}}{2d}\right)\left(\frac{1}{\nu t}\right)\left(\frac{\Delta^{1-d/2}}{\nu^{d/2} t}\right). \end{aligned} \tag{44}$$

For $d > 2$ one then finds that the ratio of $\nu E(t)$ to $\partial_t \langle h\rangle$ is

$$\frac{\nu E(t)}{(1/t)} \sim \frac{\Delta^{1-d/2}}{\nu^{d/2} t} = (t_2/t). \tag{45}$$

Therefore we have the result that for $d > 2$ in the regime $t_1 \ll t \ll t_2$ the nonlinear term $\nu(\nabla h)^2$ is highly relevant to the dynamic evolution of the system. The nontrivial result for the interface evolution in this time regime that was derived in Sec. IV, Eq. (35), may consequently be termed a strong-coupling result.

Analysis of Eq. (44) for $d < 2$ reveals that the nonlinear term is irrelevant for $t \gg t_1$ (actually it is irrelevant for all times, since for $t \ll t_1$ we have already shown that the EW results are valid). This will be demonstrated explicitly in the next section when we study a simplified version of Eq. (4) which is valid for the region $t \gg t_1$.

The case of $d = 2$ is marginal in the sense that the ratio of $\nu E(t)$ to $\partial_t\langle h\rangle$ is $O(1/\nu t)$. The cutoff Δ does not now determine the relevance of the nonlinearity. However, if ν is taken to be "very small," one may have an appreciable time regime where the nonlinearity is relevant. Investigation of this more subtle point is in progress.

VII. A SIMPLIFIED THEORY

In this section we shall motivate a simplified version of Eq. (4) which is applicable for late times. The exact results derived above indicate that the fluctuations of the interface decay for late times. Therefore the field $\tilde{h} \equiv h - \langle h\rangle$ will be much smaller than unity in the asymptotic regime [14]. We shall concentrate on the regime $t \gg t_1$. In this case $\langle h\rangle = \ln(t/t_1)$. Rewriting Eq. (4) in terms of $\tilde{h}$ we have

$$\begin{aligned} \partial_t \tilde{h} &= \nu\nabla^2\tilde{h} + \nu(\nabla\tilde{h})^2 - \frac{1}{t}[1-\exp(-\tilde{h})] \\ &\quad + \frac{(t_1\eta - 1)}{t}\exp(-\tilde{h}). \end{aligned} \tag{46}$$

We have chosen to write the equation in the above form so that the effective noise in the equation has zero mean [one can easily show that for the distribution in Eq. (3) $\langle\eta\rangle = 1/t_1$]. We prefer at this stage to replace this noise $(\eta - 1/t_1)$ by a Gaussian distributed noise ξ with zero mean and correlator,

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = (\Delta/t_1^2)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') , \tag{47}$$

which is matched to the correlator of $(\eta - 1/t_1)$. As we mentioned before, the results of the previous sections are unaffected by changing the distribution Eq. (3) to either a uniform or bimodal form. We therefore expect universality with respect to changes in the noise distribution, and can use the Gaussian noise defined above with confidence. The use of the Gaussian distribution is purely for convenience, not from necessity.

For $t \gg t_1$, the interface width as calculated in the previous sections is always much smaller than unity (as long as $\nu \ll 1/\Delta$) which implies that $\tilde{h} \ll 1$ in this regime. We may therefore expand the exponential terms in Eq. (46). Retaining the leading orders we have

$$\partial_t\tilde{h} = \nu\nabla^2\tilde{h} + \nu(\nabla\tilde{h})^2 - \frac{\tilde{h}}{t} + \frac{t_1}{t}\xi. \tag{48}$$

We expect this equation to be a valid description of the original model for all d as long as $t \gg t_1$.

For very long times, we may also discard the nonlinear term. This is consistent with the calculation of the preceding section. The resulting linear equation may be easily solved to yield

$$\tilde{h}(\mathbf{x},t) = (t_1/t)\int d^dy \int_0^t dt'\ g(\mathbf{x}-\mathbf{y},t-t')\xi(\mathbf{y},t'). \tag{49}$$

The squared interface width $W^2(t)$ is equal to $\langle\tilde{h}^2\rangle$ and may be evaluated from Eq. (49). We find

$$W^2(t) = (\Delta/t^2)\int_\Delta^t dt'\ (8\pi\nu t')^{-d/2}. \tag{50}$$

Performing the t' integral then yields the asymptotic result

$$W^2(t) = \begin{cases} \frac{2}{(d-2)}\Delta(8\pi\nu)^{-d/2}t^{-(d+2)/2} & 0<d<2 \\ \Delta(8\pi\nu)^{-1}t^{-2}\ln(t/\Delta) & d=2 \\ \frac{2}{(2-d)}\Delta^{(4-d)/2}(8\pi\nu)^{-d/2}t^{-2} & d>2 . \end{cases} \tag{51}$$

These results are precisely those obtained in the deep asymptotic regime, cf. Eqs. (25), (39), and (41). One may feel cheated at this stage since we seem to have derived most of the results of the exact calculation from a very simple linearization scheme. However, the results obtained are for the very large time regime. For $d < 2$ this regime extends all the way back to $t \sim t_1$ as we have seen. However, for $d > 2$, there is an arbitrarily large regime (in the sense of taking Δ as small as one wishes) separating the regimes associated with $t \ll t_1$ (which is just EW growth) and $t \gg t_2$ (which is EW growth with a $1/t^2$ prefactor for the mean square fluctuations). The linear equation is unable to capture this nontrivial regime—the nonlinear term $(\nabla\tilde{h})^2$ is essential. We have proved the overriding importance of this term in the preceding section for $d > 2$ in the regime $t_1 \ll t \ll t_2$.

Before discussing this issue any further we wish to direct our attention away from Eq. (48) to a related model. The transformation

$$\theta = (t/t_1)\tilde{h} \tag{52}$$

provides us with the equation

$$\partial_t \theta = \nu \nabla^2 \theta + \frac{\nu t_1}{t} (\nabla \theta)^2 + \xi . \tag{53}$$

Surprisingly we have ended up with an equation which is very closely related to the original KPZ equation, Eq. (1), the difference being in the amplitude of the nonlinear term having a factor proportional to t^{-1}. Again, we stress that this equation for θ is a valid description of our original model for $t \gg t_1$. The above calculation for the linearized equation for $\tilde{h}$ may be applied to Eq. (53). The "interface" fluctuations of θ are defined by

$$\mathcal{W}(t) = [\langle \theta^2 \rangle]^{1/2} = (t/t_1) W(t). \tag{54}$$

We see from Eq. (51) that the results for $\mathcal{W}(t)$ for very late times are simply the EW results for an interface evolving under the linear diffusion equation. This is reasonable considering the nonlinearity has an amplitude proportional to $1/t$. We also see that this EW behavior is valid for the entire regime $t \gg t_1$ for $d < 2$. However, from the analysis in previous sections we have the remarkable result that for $d > 2$ there exists a scaling regime for $t_1 \ll t \ll t_2$ for which the nonlinear term is strongly relevant. Combining Eqs. (35) and (54) we have for $t_1 \ll t \ll t_2$

$$\mathcal{W}(t) \sim \frac{\Delta^{1/d}}{\nu^{1/2} t_1} t^{\beta} , \tag{55}$$

where

$$\beta(d) = \frac{(d-2)}{2d} . \tag{56}$$

We refer the reader to Ref. [15] where simple scaling and symmetry arguments are used to study the KPZ equation with nonlinear coupling $\lambda \sim \lambda_0 t^{-\alpha}$. They find that the critical dimension of the model is shifted according to $d_c = 2 - 4\alpha$. In the present case, $\alpha = 1$, which implies that for all $d > -2$ we are above d_c and can expect an asymptotically smooth surface, as indeed was found above. The interesting behavior in our model occurs for "intermediate" times (although t_2 may be made extremely large by reducing Δ) for $d > 2$, which is not accessible from the simple arguments presented in [15].

VIII. CONCLUSIONS

We have studied a recently proposed model of interface growth which allows exact solution. The model is physically distinct from more common interface models, such as the KPZ equation, in that the fluctuations in the interface asymptotically decay in time. This "reality gap" can be eliminated though, by studying the asymptotic behavior of the field $\theta = (t/t_1)(h - \langle h \rangle)$. This new field satisfies the KPZ equation, (1), with a time-dependent coupling which is proportional to t_1/t. All results obtained for the original model, Eq. (2), may be applied to the growth equation for θ for times $t \gg t_1$.

The main quantities calculated in this paper are the interface width $W(t)$ and the interface roughness $E(t)$. For $d \leq 2$, $W(t)$ has a simple decaying form throughout the entire region $t \gg t_1$, and $E(t)$ is irrelevant. The main result of this paper is the behavior of these quantities for $d > 2$. We have found the existence of a region $t_1 \ll t \ll t_2$ within which $W(t)$ decays with nontrivial, d-dependent exponents, and where $E(t)$ is strongly relevant. For $t \gg t_2$, $W(t)$ decays as $1/t$ for all $d > 2$ and $E(t)$ becomes irrelevant.

In terms of the dynamics of θ, we find from the abovementioned result that the rms fluctuations in θ grow as t^{β} in the intermediate-time regime defined above, for $d > 2$, where $\beta = (d-2)/2d$ is a nontrivial, strong-coupling exponent.

It would be of interest to study the RG properties of Eq. (53) in order to explicitly see the emergence of the strong-coupling fixed point for intermediate times. Such a study requires the application of dynamic RG for preasymptotic times, rather than the usual aysmptotic analysis [5]. This is a nontrivial task in general, although examples are known where universality and scaling have been rigorously demonstrated using RG for intermediate times [16].

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